Global well-posedness for the Oberbeck-Boussinesq system

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Abstract

This work deals with a study on regularities of global-in-time solution, its uniform stability and uniqueness for the incompressible Oberbeck-Boussinesq system.

1 Introduction

The Oberbeck–Boussinesq approximation is a mathematical model of a stratified fluid flow, where the fluid is assumed to be incompressible and yet convecting a diffusive quantity. The diffusive quantity is identified with the deviation of temperature from its equilibrium value. A model that describes this dynamics of a viscous incompressible fluid with heat exchanges on a bounded domain can be represented by the following Oberbeck–Boussinesq system:

$$\left\{\begin{array}{rcll} \partial_t \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - \mu\Delta\boldsymbol{u} + \nabla p & = & \alpha\theta\boldsymbol{g} + \boldsymbol{j}, & \text{in } \Omega\times(0,T) \\ \partial_t\theta + (\boldsymbol{u}\cdot\nabla)\theta - \kappa\Delta\theta & = & f, & \text{in } \Omega\times(0,T) \\ \operatorname{div}\boldsymbol{u} & = & 0 & \text{in } \Omega\times(0,T) \end{array}\right. \tag{1.1}$$

supplemented with homogeneous Dirichlet boundary conditions and initial data

$$\begin{aligned} \boldsymbol{u}(x,t) = 0 \quad &\text{and} \quad \theta(x,t) = 0 \text{ on } \partial\Omega\times(0,T), \\ \boldsymbol{u}(x,t) = \boldsymbol{u}_0 \quad &\text{and} \quad \theta(x,t) = \theta_0 \text{ in } \Omega, \end{aligned}$$

where $\Omega \subseteq \mathbb{R}^3$ is bounded-open set, with smooth boundary $\partial\Omega$ and $0 < T \leq \infty$. Moreover, $\boldsymbol{u} : \Omega\times(0,T) \to \mathbb{R}^3$, $\theta : \Omega\times(0,T) \to \mathbb{R}$ and $p : \Omega\times(0,T) \to \mathbb{R}$ denote the velocity vector, the temperature and the pressure at time $t \in [0,T)$ and at point $x \in \Omega$ respectively. The functions $\boldsymbol{g}, \boldsymbol{j} : \Omega\times(0,T) \to \mathbb{R}^3$ and

$f : \Omega \times (0,T) \to \mathbb{R}$ represent external sources, $\mu > 0$ is the viscosity of fluid and κ is the thermal diffusivity.

The variational formulation of (1.1) consist in finding $\boldsymbol{u}$ and θ in the class $\boldsymbol{u} \in L^2(0,T;\boldsymbol{V})$ and $\theta \in L^2(0,T;H_0^1(\Omega))$ for all $0 < T \leq +\infty$, such that,

$$\begin{aligned} &\frac{d}{dt}(\boldsymbol{u},\varphi) + \mu(\nabla \boldsymbol{u}, \nabla\varphi) + ((\boldsymbol{u}\cdot\nabla)\boldsymbol{u},\varphi) = \alpha(\theta \boldsymbol{g},\varphi) + (\boldsymbol{j},\varphi) \\ &\frac{d}{dt}(\theta,\phi) + \kappa(\nabla\theta,\nabla\phi) + ((\boldsymbol{u}\cdot\nabla)\theta,\phi) = (f,\phi) \\ &\boldsymbol{u}(0) = \boldsymbol{u}_0,\ \theta(0) = \theta_0 \end{aligned} \tag{1.2}$$

We denote by B the Laplace operator $-\Delta$ with the standard domain $D(B) = H_0^1(\Omega) \cap H^2(\Omega)$ and we denote by A the Stokes operator $A = P\Delta$, here P is the Helmholtz projection, with the domain $D(A) = \boldsymbol{V} \cap \boldsymbol{H}^2(\Omega)$. The following theorem is the main result of this work.

Theorem 1. *Let* $(\boldsymbol{u},\theta) \in L^\infty(0,T;D(A^{1/4})) \cap L^2(0,T;D(A^{3/4})) \times L^\infty(0,T;D(B^{1/4})) \cap L^2(0,T;D(B^{3/4}))$ *be a solution of* (1.2) *with the initial conditions* $\boldsymbol{u}(0) = \boldsymbol{u}_0 \in D(A^{1/4}), \theta(0) = \theta_0 \in D(B^{1/4})$ *satisfying*

$$\int_0^T \left(\|A^{3/4}\boldsymbol{u}(t)\|^2 + \|A^{1/2}\boldsymbol{u}(t)\|^4 \right) dt \quad < \quad \infty.$$

Then to given $\epsilon > 0$ *there exists* $\delta > 0$ *such that if* $(\boldsymbol{v}_0, \vartheta_0) \in D(A^{1/4}) \times D(B^{1/4}), P\boldsymbol{j}_1 \in L^2(0,\infty;\boldsymbol{H}), f \in L^2(0,\infty;L^2(\Omega))$ *are functions satisfying*

$$\|A^{1/4}(\boldsymbol{u}_0 - \boldsymbol{v}_0)\| + \|B^{1/4}(\theta_0 - \vartheta_0)\| + \int_0^T \|P\boldsymbol{j}(t) - P\mathbf{j}_1(t)\|^2\, dt + \int_0^T \|f(t) - f_1(t)\|^2\, dt \quad < \quad \delta$$

then there exits a unique strong solution $(\boldsymbol{v},\vartheta)$ *of the problem* (1.2) *with the data* $(\boldsymbol{v}_0,\vartheta_0)$ *and* $\boldsymbol{j}_1$ *(instead of* $(\boldsymbol{u}_0,\theta_0)$ *and* $\boldsymbol{j}$*) on the interval* $(0,+\infty)$*, satisfying*

$$\|A^{1/4}(\boldsymbol{v}(t) - \boldsymbol{u}(t))\|^2 + \int_0^t \|A^{3/4}(\boldsymbol{v}(s) - \boldsymbol{u}(s)\|^2\, ds \quad \leq \quad \epsilon$$

and

$$\|B^{1/4}(\vartheta(t) - \theta(t))\|^2 + \int_0^t \|B^{3/4}(\vartheta(s) - \theta(s)\|^2\, ds \quad \leq \quad \epsilon$$

for all $t \in (0,T)$.

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