Euler-Lagrangian approach to stochastic Euler equations in Sobolev Spaces

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In this talk i establish the equivalence between Lagrangian and classical formulations for the stochastic incompressible Euler equations, the proof is based in Ito-Wentzell-Kunita formula and stochastic analysis techniques. Moreover, we prove a local existence result for the Lagrangian formulation in suitable Sobolev Spaces.

We study a Lagragian formulation (following [3], [6] and [12]) of the incompressible Euler equations on a domain $\mathbb{T}^d$. The Euler equations with transport noise model the flow of an incompressible inviscid fluid and are (classically) formulated in terms of a divergence–free vector field u (i.e. $\nabla \cdot u = 0$) as follows:

$$du_t + (u_t \cdot \nabla u_t + \nabla p_t)dt + \sum_k \mathcal{L}^*_{\sigma^k} u_t \ \circ dW^k_t = 0 \tag{1}$$

where p is a scalar potential representing internal pressure, $\mathcal{L}^*_{\sigma^j} u := \sigma^j \cdot \nabla u + \left(\nabla \sigma^j\right)^* u$ ($\mathcal{L}$ is the Lie derivative), W^k_t is a Wiener process and the integration is in the Stratonovich sense. The divergence-free condition reflects the incompressibility constraint. Equations related to fluid dynamics with multiplicative noise appeared in several other works, see for instance [1], [5],[6], [7], [8] and many others.

The main topic of this work, namely the Euler-Lagrangian formulation, called also Constantin-Iyer representation after [3], [4], among related works, see for instance [2], [9], [11]. First we show the Euler-Lagrangian formulation is equivalent to the stochastic Euler equations (1), see Proposition 1, the proof is based in Ito-Wentzell-Kunita formula and stochastic analysis techniques. We point that in [6] the authors show that the Lagrangian formulation verifies necessarily the equation (1), for $d = 3$, using the vorticity equation. We show that both formulations are equivalent for any dimension. Using this formulation we prove a local in time existence result for solutions in $C^0([0,T]\,; (H^s(\mathbb{T}^d)^d))$ with $s > \frac{d}{2} + 1$, new for equation (1).

The main result are the following theorems.

Theorem 1. *Assume that u is $C^{3,\alpha}$-continuos semimartingale. Then u is solution of the equation (1) if and only if verifies the Lagrangian formulation*

$$dX_t = \textstyle\sum_j \sigma^j(X_t) \circ dW^j_t + u_t(X_t)dt \tag{2}$$

$$u_t(x) = \mathbb{P}\left[(\nabla A_t)^* u_0(A_t)\right](x), \tag{3}$$

where $*$ means the transposition of matrices and denote the back-to-labels map A by setting $A\left(\cdot, t\right) = X^{-1}\left(\cdot, t\right)$.

Theorem 2. *If $d \geq 2$, $s > \frac{d}{2} + 1$ and $u_0 \in H^s$ is divergence free then there exists $T(\omega) > 0$, such that the systems*

$$\partial_t v + (\tilde{u} \cdot \nabla) v = 0 \tag{4}$$

$$u_t(x) = \mathbb{P}\left[(\nabla A_t)^* u_0(A_t) \right](x), \tag{5}$$

with initial condition $u(x,0) = v(x,0) = u_0(x)$ has solution $u \in C^0([0,T]\,;(H^s(\mathbb{T}^d)^d))$

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