Aula 13

Matriz Mudança de base

Sejam $E = (\vec{e_1}, \vec{e_2}, \vec{e_3})$ e $F = (\vec{f_1}, \vec{f_2}, \vec{f_3})$ duas bases e $(\vec{v})_E = (x_1, x_2, x_3)$ e $(\vec{v})_F = (y_1, y_2, y_3)$. Qual a relação entre $(\vec{v})_F$ e $(\vec{v})_E$?

- $\vec{v} = x_1 \vec{e_1} + x_2 \vec{e_2} + x_3 \vec{e_3}$
  $\vec{v} = y_1 \vec{f_1} + y_2 \vec{f_2} + y_3 \vec{f_3}$

Os vetores da base F se escrevem como C.L. de vetores da base E, digamos

$\vec{f_1} = a_1 \vec{e_1} + a_2 \vec{e_2} + a_3 \vec{e_3}$ $\qquad (\vec{f_1})_E = (a_1, a_2, a_3)$

$\vec{f_2} = b_1 . \vec{e_1} + b_2 . \vec{e_2} + b_3 \vec{e_3}$ $\qquad (\vec{f_2})_E = (b_1, b_2, b_3)$

$\vec{f_3} = c_1 . \vec{e_1} + c_2 \vec{e_2} + c_3 \vec{e_3}$ $\qquad (\vec{f_3})_E = (c_1, c_2, c_3)$

$$\vec{v} = y_1 . \vec{f_1} + y_2 \vec{f_2} + y_3 \vec{f_3}$$

$$= y_1 . (a_1 \vec{e_1} + a_2 \vec{e_2} + a_3 \vec{e_3}) + y_2 . (b_1 \vec{e_1} + b_2 \vec{e_2} + b_3 \vec{e_3}) + y_3 . (c_1 \vec{e_1} + c_2 \vec{e_2} + c_3 \vec{e_3})$$

$$= \vec{e_1} (a_1 y_1 + b_1 y_2 + c_1 y_3) + \vec{e_2} (a_2 y_1 + b_2 y_2 + c_2 y_3) + \vec{e_3} (a_3 y_1 + b_3 y_2 + c_3 y_3).$$

Mas $\vec{v} = \vec{e_1} . x_1 + \vec{e_2} . x_2 + \vec{e_3} . x_3$

$$
\begin{cases} x_1 = a_1 y_1 + b_1 y_2 + c_1 y_3 \\ x_2 = a_2 y_1 + b_2 y_2 + b_3 y_3 \\ x_3 = a_3 y_1 + b_3 y_2 + c_3 y_3 \end{cases} \iff \underbrace{\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}}_{A} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}
$$

A: É a matriz que relaciona as coordenadas da base F para base E.

$A = M_E^F$ no qual as colunas de M_E^F são $(\vec{f_1})_E, (\vec{f_2})_E, (\vec{f_3})_E$

$$
\underbrace{\begin{pmatrix} (\vec{f_1})_E & (\vec{f_2})_E & (\vec{f_3})_E \\ & & \\ & & \end{pmatrix}}_{M_E^F} \cdot (\vec{v})_F = (\vec{v})_E
$$

Valem as seguintes fórmulas

(i) $M_E^E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

(ii) Sejam E, F e G três bases. Então vale

$$M_G^F \cdot M_F^E = M_G^E$$

(iii) Caso particular de (ii)

$$M_E^F \cdot M_F^E = M_E^E = I_3$$

$$\Rightarrow \underbrace{(M_F^E)^{-1}}_{\text{inversa}} = M_E^F$$

Mudança de Sist. de coord.

Sejam $S_1 = (O; E)$ $\quad E = (\vec{e_1}, \vec{e_2}, \vec{e_3})$

$S_2 = (O'; F)$ $\quad F = (\vec{f_1}, \vec{f_2}, \vec{f_3})$

$(P)_{S_1} = (x_1, x_2, x_3)$. Qual é a rel. entre $(P)_{S_1}$ e $(P)_{S_2}$?

$(P)_{S_2} = (y_1, y_2, y_3)$

$(\vec{OP})_{S_1} = (x_1, x_2, x_3)$. Mas $\vec{OP} = \vec{OO'} + \vec{O'P}$

$(\vec{OP})_{S_1} = (\vec{OO'})_{S_1} + (\vec{O'P})_{S_1}$

$(\vec{O'P})_{S_2} = (y_1, y_2, y_3)$

$(\vec{OP})_{S_1} = M_{S_1}^{S_2} (\vec{O'P})_{S_2} + (\vec{OO'})_{S_1}$

$\therefore \quad (P)_{S_1} = M_{S_1}^{S_2} (P)_{S_2} + (O')_{S_1}$

Analog, $\quad (P)_{S_2} = M_{S_2}^{S_1} (P)_{S_1} + (O)_{S_2}$

Exercício: Seja $S_1 = \{A; \vec{AB}, \vec{AA'}, \vec{AF}\}$ na escadinha.

$S_2 = \{C; \vec{CD}, \vec{CB}, \vec{CA'}\}$

(a) Ache diretamente a eq. do plano CA'A em relação ao sist. S_1 e em rel. ao sist. S_2.

(b) Ache, usando mudança de base / sist. de coord., a eq. geral do plano CA'A em relação a S_2 a partir da eq. geral em rel. ao sist. S_1.

Sol

(a) $\vec{AA'}, \vec{AC}, \vec{AX}$ são L.D. Se $(X)_{S_1} = (x, y, z)$

temos que

$$\det\begin{pmatrix} 0 & 1 & x \\ 1 & 0 & y \\ 0 & \frac{1}{2} & z \end{pmatrix} = 0 \iff x - 2z = 0$$ eq. geral do plano CA'A em rel. a S_1

$$\downarrow \quad (\vec{AA'})_{S_1} \quad \downarrow \quad (\vec{AC})_{S_1} \quad \downarrow \quad (\vec{AX})_{S_1}$$

$\vec{CA}$, $\vec{CA'}$, $\vec{CX}$ são L.D. Se $(X)_{S_2} = (x', y', z')$

$$\det\begin{pmatrix} 2 & 0 & x' \\ 1 & 0 & y' \\ 0 & 1 & z' \end{pmatrix} = 0 \iff x' - 2y' = 0$$ eq. geral do plano CA'A em rel. a S_2.

$$\downarrow \quad (\vec{CA})_{S_2} \quad \downarrow \quad (\vec{CA'})_{S_2} \quad \downarrow \quad (\vec{CX})_{S_2}$$

(b) $(P)_{S_1} = (x, y, z)$
$(P)_{S_2} = (x', y', z')$ $\Rightarrow (P)_{S_1} = M_{S_1}^{S_2} (P)_{S_2} + (C)_{S_1}$

$(C)_{S_1} = (\vec{AC})_{S_1} = (1, 0, 1/2)$

$(\vec{CD})_{S_1} = (-1/2, 0, 0)$
$(\vec{CB})_{S_1} = (0, 0, -1/2)$
$(\vec{CA'})_{S_1} = (-1, +1, -1/2)$

$$\therefore M_{S_1}^{S_2} = \begin{pmatrix} -1/2 & 0 & -1 \\ 0 & 0 & 1 \\ 0 & -1/2 & -1/2 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1/2 & 0 & -1 \\ 0 & 0 & 1 \\ 0 & -1/2 & -1/2 \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 1/2 \end{pmatrix} = \begin{pmatrix} \frac{-x'}{2} - z' + 1 \\ 0 + z' + 0 \\ \frac{-y'}{2} - \frac{z'}{2} + \frac{1}{2} \end{pmatrix}$$

$$\therefore x - 2z = 0 \iff \left(\frac{-x'}{2} - z' + 1\right) - 2\cdot\left(\frac{-y'}{2} - \frac{z'}{2} + \frac{1}{2}\right) = 0 \iff x' - 2y' = 0.$$

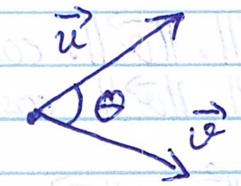
Aula 14

Relações Métricas e medida de Ângulos

Definição: dados dois vetores $\vec{u}$ e $\vec{v}$, o produto ESCALAR $\vec{u}$ por $\vec{v}$ é o nº real definido

$$\langle \vec{u}, \vec{v} \rangle = \vec{u} \cdot \vec{v} = \|\vec{u}\| \cdot \|\vec{v}\| \cdot \cos\theta$$

no qual $0 \leq \theta \leq 180°$ é o ângulo entre os vetores $\vec{u}$ e $\vec{v}$



Propriedades:

(i) $\langle \vec{u}, \vec{u} \rangle = \|\vec{u}\|^2, \forall \vec{u}$
(ii) $\langle \vec{u}, \vec{v} \rangle = \langle \vec{v}, \vec{u} \rangle \quad \forall \vec{v}, \vec{u}$
(iii) Se $\vec{u} \neq 0$ e $\vec{v} \neq 0$ e $\vec{u} \cdot \vec{v} = 0 \Rightarrow \theta = 90°$ e $\therefore \vec{u} \perp \vec{v}$
(iv) $(\lambda \vec{u}) \cdot \vec{v} = \lambda \cdot \vec{u} \cdot \vec{v} = \vec{u} \cdot (\lambda \vec{v}) \quad \forall \lambda \in \mathbb{R}, \forall \vec{u}, \vec{v}$.
(v) $(\vec{u} + \vec{v}) \cdot \vec{w} = \vec{u} \cdot \vec{w} + \vec{v} \cdot \vec{w}$.

Objetivo: como calcular $\vec{u} \cdot \vec{v}$?

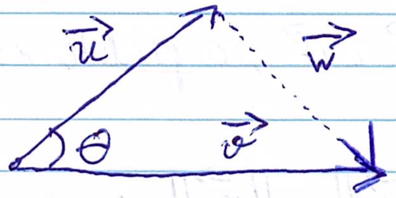
Definição: Uma base ortonormal é uma base onde os vetores são unitários e dois a dois ortogonais. Indicamos uma base ortonormal p/ o espaço por

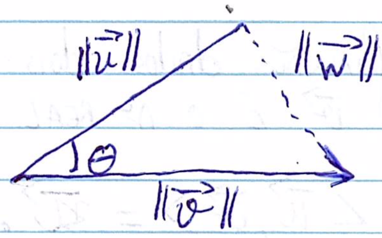
$$\{\vec{i}, \vec{j}, \vec{k}\} \quad (\text{são L.I.})$$

Note que $\vec{i} \cdot \vec{j} = 0$ e $\vec{k} \cdot \vec{j} = 0$. , $\|\vec{i}\| = \|\vec{j}\| = \|\vec{k}\| = 1$.
$\vec{i} \cdot \vec{k} = 0$

Agora, sejam $(\vec{u})_\beta = (x_1, x_2, x_3)$ e $(\vec{v})_\beta = (y_1, y_2, y_3)$

para $\beta = \{\vec{i}, \vec{j}, \vec{k}\}$





Pela Lei dos cossenos

$$\|\vec{w}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2 . \|\vec{u}\| . \|\vec{v}\| . \cos\theta$$
$$\|\vec{v} - \vec{u}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2 . \|\vec{u}\| . \|\vec{v}\| . \cos\theta$$

$$\therefore \quad \|\vec{u}\| . \|\vec{v}\| . \cos\theta = \frac{\|\vec{u}\|^2 + \|\vec{v}\|^2 - \|\vec{v} - \vec{u}\|^2}{2}$$

$$\vec{u} \cdot \vec{v} = \frac{1}{2}\left(\|\vec{u}\|^2 + \|\vec{v}\|^2 - \|\vec{v} - \vec{u}\|^2 \right)$$

Problema: como calcular $\|\vec{u}\|$, $\|\vec{v}\|$, $\|\vec{u} - \vec{v}\|$ em função de β.

Seja $(\vec{u})_\beta = (x_1, x_2, x_3)$ e $(\vec{v})_\beta = (y_1, y_2, y_3)$ p/ $\beta = \{\vec{i}, \vec{j}, \vec{k}\}$

$$\vec{u} = x_1 . \vec{i} + x_2 . \vec{j} + x_3 . \vec{k}$$

$\Downarrow$ Teorema de Pitágoras

$$\|\vec{u}\|^2 = \|x_1 . \vec{i}\|^2 + \|x_2 . \vec{j}\|^2 + \|x_3 . \vec{k}\|^2$$
$$= x_1^2 . \|\vec{i}\|^2 + x_2^2 . \|\vec{j}\|^2 + x_3^2 . \|\vec{k}\|^2 \quad , \beta \text{ ortonormal}$$
$$= (x_1)^2 + (x_2)^2 + (x_3)^2$$

$$\therefore \quad \|\vec{u}\| = \left((x_1)^2 + (x_2)^2 + (x_3)^2 \right)^{\frac{1}{2}}$$

$$\|\vec{v}\| = \left((y_1)^2 + (y_2)^2 + (y_3)^2 \right)^{\frac{1}{2}}$$

$$\|\vec{u} - \vec{v}\| = \left((x_1-y_1)^2 + (x_2-y_2)^2 + (x_3-y_3)^2 \right)^{\frac{1}{2}}$$

Substituindo na fórmula temos

$$\vec{u}\cdot\vec{v} = \frac{1}{2}\left(x_1^2 + x_2^2 + x_3^2 + y_1^2 + y_2^2 + y_3^3 - \left[(x_1-y_1)^2 + (x_2-y_2)^2 + (x_3-y_3)^2 \right] \right)$$

$$= x_1 \cdot y_1 + x_2 \cdot y_2 + x_3 y_3$$

$$\vec{u}\cdot\vec{v} = x_1 \cdot y_1 + x_2 \cdot y_2 + x_3 \cdot y_3$$

no qual $(\vec{u})_\beta = (x_1, x_2, x_3)$, $(\vec{v})_\beta = (y_1, y_2, y_3)$ e $\beta = \{\vec{i}, \vec{j}, \vec{k}\}$.

Exercícios

(a) Calcule o ângulo entre $\overrightarrow{AF'}$ e $\overrightarrow{AC'}$

(b) " um vetor $\vec{w} \perp$ ao plano ADF'

(c) Calcule a distância do pto E' ao plano ADF'.

sol.

(a) Considere $S = \left(A; \underbrace{\frac{1}{4}\cdot\overrightarrow{AB}, \frac{1}{4}\overrightarrow{AF}, \frac{1}{2}\overrightarrow{AA'}}_{\beta} \right)$

• β é ortonormal

$(\overrightarrow{AF'})_S = (0,4,2) \Rightarrow \|\overrightarrow{AF'}\| = \sqrt{20}$

$(\overrightarrow{AC'})_S = (4,2,2) \qquad \|\overrightarrow{AC'}\| = \sqrt{24}$

$$\overrightarrow{AF'} \cdot \overrightarrow{AC'} = \|\overrightarrow{AF'}\| . \|\overrightarrow{AC'}\| . \cos\theta$$

$$0.4 + 4.2 + 2.2 = \sqrt{20} . \sqrt{24} . \cos\theta$$

$$\therefore \cos\theta = \frac{12}{\sqrt{20}\cdot\sqrt{24}} = \frac{3}{\sqrt{30}} = \frac{\sqrt{30}}{10}$$

$$\therefore \theta = \text{ARCCOS}\left(\frac{\sqrt{30}}{10}\right).$$

(b)

$$\vec{w} \perp \text{plano } ADF' \iff \begin{array}{l} \vec{w} \perp \overrightarrow{AF'} \\ \vec{w} \perp \overrightarrow{AD} \end{array}$$

$$(\vec{w})_\beta = (z_1, z_2, z_3)$$

$$(\overrightarrow{AF'})_s = (0,4,2) \qquad \vec{w}\cdot\overrightarrow{AF'} = z_1\cdot 0 + 4\cdot z_2 + 2\cdot z_3 = 0$$

$$(\overrightarrow{AD})_s = (2,2,0) \qquad \vec{w}\cdot\overrightarrow{AD} = z_1\cdot 2 + 2\cdot z_2 + 0\cdot z_3 = 0$$

$$\iff \begin{array}{ll} 4z_2 + 2z_3 = 0 & \Rightarrow z_3 = -2z_2 \\ 2z_1 + 2z_2 = 0 & \Rightarrow z_1 = -z_2. \end{array}$$

$$(\vec{w})_\beta = (z_1,z_2,z_3) = (-z_2, z_2, -2z_2) = z_2\cdot(-1, \ , -2) \quad \boxed{z_2 = 1}$$

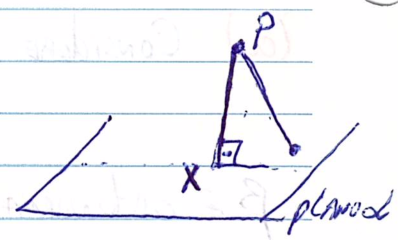
$$\vec{w} = -1\cdot\vec{i} + 1\cdot\vec{j} - 2\cdot\vec{k} = -\frac{1}{4}\overrightarrow{AB} + \frac{1}{4}\overrightarrow{AF} - \overrightarrow{AA'}.$$

(c)

$$X = A + t_1\cdot\overrightarrow{AF'} + t_2\cdot\overrightarrow{AD}$$

$$X = E + s\cdot\vec{v} \qquad \vec{v} \perp \text{plano } ADF'$$

$$(\vec{v})_\beta = (-1, 1, -2)$$



Eq. geral do plano ADF'

$$\det\begin{pmatrix} x & 0 & 2 \\ y & 4 & 2 \\ z & 2 & 0 \end{pmatrix} = 0 \iff y - x - 2z = 0$$

Eq. paramétrica de $X = E + s\cdot\vec{v}$ $\quad (E)_s = (2,4,0)$

$$\begin{cases} x = 2 - s \\ y = 4 + s \\ z = -2s \end{cases}$$

$$\Rightarrow s = -\frac{1}{3} \Rightarrow \|\overrightarrow{EX}\| = \frac{\sqrt{6}}{3}.$$

$$(X)_s = \frac{1}{3}\cdot(7,11,2) \longrightarrow (\overrightarrow{XE})_s = \left(-\frac{1}{3}, \frac{1}{3}, -\frac{2}{3}\right)$$