Aula 20

Caso geral: Cônicas

Exemplo: $f(x,y) \doteq \frac{3}{2}x^2 + 3yx + \frac{3}{2}y^2 + x - y = 0$

1º passo) Escrever na forma matricial

$$f(x,y) \doteq (x\ y)\underbrace{\begin{pmatrix} 3/2 & 3/2 \\ 3/2 & 3/2 \end{pmatrix}}_{A}\begin{pmatrix} x \\ y \end{pmatrix} + 2.\underbrace{\begin{pmatrix} 1/2 & -1/2 \end{pmatrix}}_{B}\begin{pmatrix} x \\ y \end{pmatrix} = 0$$

2º passo) Achando o centro $(C)_S = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$ tq $AC = -B^t$

isto é, $\begin{pmatrix} 3/2 & 3/2 \\ 3/2 & 3/2 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} -1/2 \\ 1/2 \end{pmatrix} \Rightarrow$ este sistema não tem solução! S.I.

E agora?

Considere o sistema $S' = \{0; \vec{u_1}, \vec{u_2}\}$ onde $\vec{u_1}$ e $\vec{u_2}$ são autovetores unitários dos autovalores da matriz A.

$$\det(A - \lambda I) = 0 \iff \begin{vmatrix} \frac{3}{2}-\lambda & \frac{3}{2} \\ 3/2 & 3/2-\lambda \end{vmatrix} = 0 \iff \left(\frac{3}{2}-\lambda\right)^2 - \left(\frac{3}{2}\right)^2 = 0$$

$\iff \lambda_1 = 0$ ou $\lambda_2 = 3$

Calculando os autovet. corresp. a $\lambda_1 = 0$. Assim $v = (a,b) \neq (0,0)$ é autovetor associado a $\lambda_1 = 0 \iff$

$$\begin{pmatrix} 3/2 & 3/2 \\ 3/2 & 3/2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff \frac{3}{2}a + \frac{3}{2}b = 0 \iff -a = +b$$

$\therefore v = (a, -a) = a.(1,-1)\ \forall a \in \mathbb{R}^*$

$\Rightarrow \vec{u_1} = \frac{(1,-1)}{\sqrt{2}}$.

Calculando os autovet. corresp. a $\lambda = 3$. Assim $v = (a,b) \neq (0,0)$ é autovetor associado a $\lambda = 3 \iff$

$$\begin{pmatrix} 3/2 - 3 & 3/2 \\ 3/2 & 3/2 - 3 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff -\frac{3}{2}a + \frac{3}{2}b = 0 \iff a = b$$

$\therefore v = a.(1,1)$ $\forall a \in \mathbb{R}^*$

$\vec{u_2} = \dfrac{(1,1)}{\sqrt{2}}$. Note que $\vec{u_1} \cdot \vec{u_2} = 0$ e $\therefore S' = \{O; \vec{u_1}, \vec{u_2}\}$ um sist. formado por uma base ortogonal.

prod. interno

3º passo) Escrevendo a eq. da cônica em relação ao sist. S'

$$\begin{pmatrix} x \\ y \end{pmatrix} = M_{S}^{S'}\begin{pmatrix} x' \\ y' \end{pmatrix} + (0)_S \iff \begin{pmatrix} x \\ y \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}}_{R}\begin{pmatrix} x' \\ y' \end{pmatrix}$$

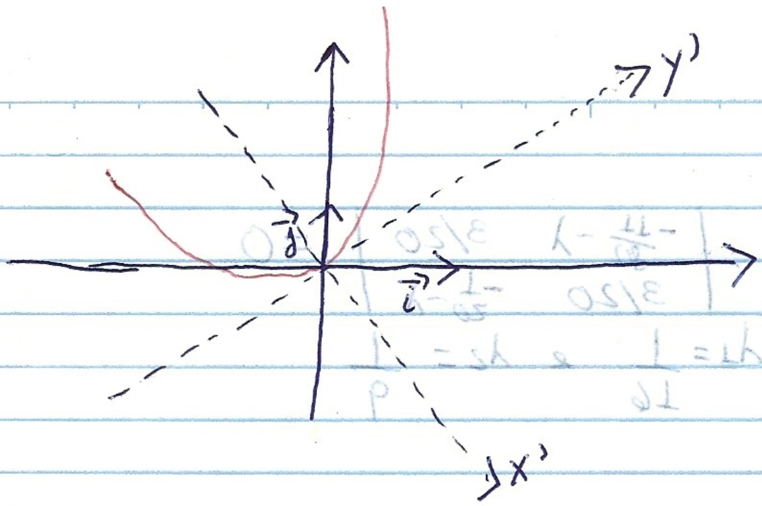
$X = R.X'$ $\therefore$
$$\begin{aligned} f(X) &= X^t A X + 2BX + K \\ &= (RX')^t A.RX' + 2BRX' + K \\ &= X'^t \underbrace{R^t A R}\, X' + 2BRX' + K, \text{ no qual} \end{aligned}$$

$$R^t A R = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} 3/2 & 3/2 \\ 3/2 & 3/2 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 3 \end{pmatrix}, \text{ e}$$

$$2B.R = 2.(1/2 \quad -1/2)\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = \left(\frac{2}{\sqrt{2}} \quad 0\right). \text{ Assim}$$

$$\begin{aligned} f(X') &= X'^t \begin{pmatrix} 0 & 0 \\ 0 & 3 \end{pmatrix} X' + (\sqrt{2} \quad 0)\, X' = 0 \\ &= (x' \quad y')\begin{pmatrix} 0 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} x' \\ y' \end{pmatrix} + (\sqrt{2} \quad 0)\begin{pmatrix} x' \\ y' \end{pmatrix} = 0 \end{aligned}$$

$$\Rightarrow f(x', y') \doteq 3(y')^2 + \sqrt{2}\, x' = 0.$$



Exemplo 2: $S=\{O;\vec{i},\vec{j}\}$

$$f(x,y) = -\frac{11}{80}x^2 + \frac{9}{40}x + \frac{3}{10}y.x - \frac{7}{10}y - \frac{1}{20}y^2 - 1 + \frac{29}{80} = 0$$

1º passo)

$$f(X) = X^t \begin{pmatrix} -\frac{11}{80} & 3/20 \\ 3/20 & -1/20 \end{pmatrix} X + 2\cdot\begin{pmatrix} \frac{9}{80} & -\frac{7}{20} \end{pmatrix} X + \left(-1 + \frac{29}{80}\right) = 0$$

2º passo) Centro

$$\begin{pmatrix} -11/80 & 3/20 \\ 3/20 & -1/20 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} -9/80 \\ 7/20 \end{pmatrix} \Rightarrow (C)_S = (3,2)$$

3º passo) Escrever a cônica em rel. ao sistema $S'=\{C;\vec{i},\vec{j}\}$

$$X = X' + C \Rightarrow f(X) = f(X'+C) = 0$$

$$\Rightarrow f(X') = X'^t A X' + f(C) = 0$$

4º passo) Fazer uma rotação do sist. S' p/ o sist. $S''=\{C;\vec{u_1},\vec{u_2}\}$ no qual $\vec{u_1}$ e $\vec{u_2}$ são autovet. de A.

Autovalores de A

$$\det(A-\lambda I)=0 \iff \begin{vmatrix} -\frac{11}{80}-\lambda & 3/20 \\ 3/20 & -\frac{1}{20}-\lambda \end{vmatrix} = 0$$

$$\iff \lambda_1 = \frac{1}{16} \quad e \quad \lambda_2 = \frac{1}{9}$$

$\lambda_1 = \frac{1}{16}$

$$\begin{pmatrix} -\frac{11}{80}-\frac{1}{16} & 3/20 \\ 3/20 & -\frac{1}{20}-\frac{1}{16} \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff -4a+3b=0$$

$$\vec{v_1} = (3,4)$$

$$\vec{u_1} = \frac{\vec{v_1}}{\|\vec{v_1}\|} = \frac{(3,4)}{5}$$

Analog. $\vec{u_2} = \frac{(-4,-3)}{5}$

5º passo -) Escrever a eq. em rel. ao sist. S'' $\quad M_{S''}^{S'}$

$$X'^t A X' + f(c) = 0 \quad , \quad X' = R X''$$

$$X' = \begin{pmatrix} 3/5 & -4/5 \\ 4/5 & 3/5 \end{pmatrix}$$

$$X''^t R^t A R X'' + f(c) = 0$$

$$R^t A R = \begin{pmatrix} 1/16 & 0 \\ 0 & -1/4 \end{pmatrix} \quad e \quad f(c) = -1$$

$$X''^t \begin{pmatrix} 1/16 & 0 \\ 0 & -1/4 \end{pmatrix} X'' - 1 = 0 \iff \left(\frac{x''}{4}\right)^2 - \left(\frac{y''}{2}\right)^2 = 1$$