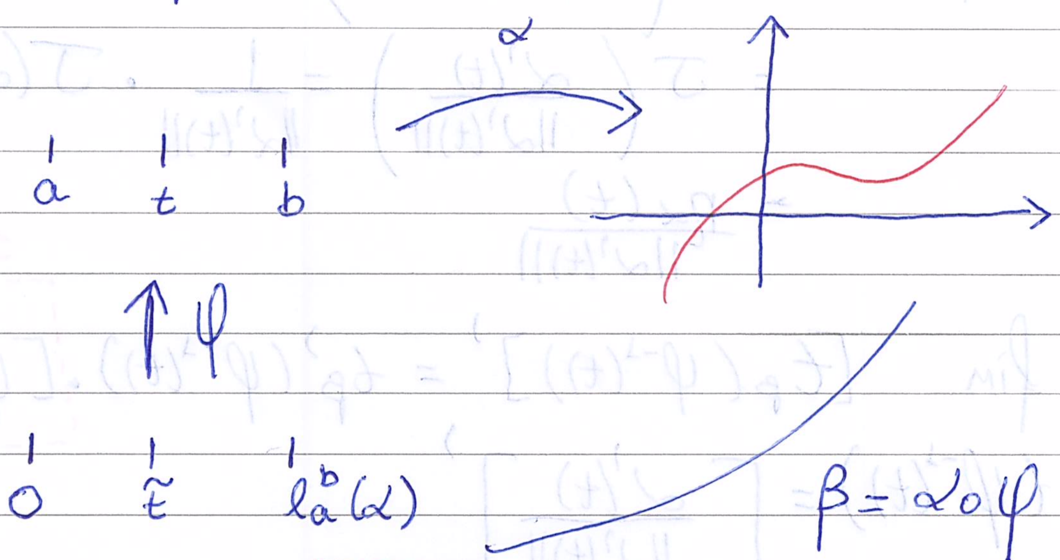
Cálculo da curvatura de uma curva regular não ppca

Seja $\alpha : I \to \mathbb{R}^2$ uma curva dif. regular não necessariamente ppca. Seja β uma reparametrização de α pelo comprimento do arco.



Lembrando que $\boxed{\varphi = s^{-1}}$ no qual $s(t) = \int_a^t \|\alpha'(u)\| du$.

Definição: $K_\alpha(t) \doteq K_\beta(\varphi^{-1}(t))$

Pergunta: como encontrar K_α em função de suas coord?

Vamos fazer algumas continhas

$$\alpha = \beta \circ s \Rightarrow \alpha'(t) = \beta'(s(t)) . s'(t)$$

$$\therefore t_\alpha(t) = t_\beta(s(t)) . \|\alpha'(t)\|.$$

$$\therefore\ t_\beta(\varphi^{-1}(t)) = \frac{t_\alpha(t)}{\|\alpha'(t)\|} = \frac{\alpha'(t)}{\|\alpha'(t)\|}$$

Agora

$$\eta_\beta(\varphi^{-1}(t)) = J\left(t_\beta(\varphi^{-1}(t))\right)$$

$$= J\left(\frac{\alpha'(t)}{\|\alpha'(t)\|}\right) = \frac{1}{\|\alpha'(t)\|}\cdot J(\alpha'(t))$$

$$= \frac{\eta_\alpha(t)}{\|\alpha'(t)\|}.$$

Por fim $\quad [t_\beta(\varphi^{-1}(t))]' = t_\beta'(\varphi^{-1}(t)) \cdot \overbrace{[\varphi^{-1}]'(t)}^{s'(t)}$

E

$$t_\beta'(\varphi^{-1}(t)) = \left[\frac{\alpha'(t)}{\|\alpha'(t)\|}\right]'$$

$$= \frac{\alpha''(t)}{\|\alpha'(t)\|} + \alpha'(t)\cdot\left[\frac{1}{\|\alpha'(t)\|}\right]'$$

portanto

$$t_\beta'(\varphi^{-1}(t)) = \frac{\alpha''(t)}{\|\alpha'(t)\|^2} + \frac{\alpha'(t)}{\|\alpha'(t)\|}\cdot\left[\frac{1}{\|\alpha'(t)\|}\right]'$$

Conclusão:

$$K_\alpha(t) \doteq K_\beta(\varphi^{-1}(t)) = \langle t_\beta'(\varphi^{-1}(t)), \eta_\beta(\varphi^{-1}(t))\rangle$$

$$= \frac{\langle t_\alpha'(t), \eta_\alpha(t)\rangle}{\|\alpha'(t)\|^3}$$

Note que K_α não depende de φ. Além do mais em coord. temos

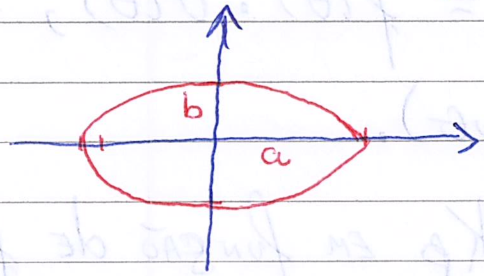
$$t_\alpha(t) = \alpha'(t) = (x'(t), y'(t)) \therefore \eta_\alpha(t) = (-y'(t), x'(t)).$$

$$\Rightarrow t_\alpha'(t) = (x''(t), y''(t))$$. Assim

$$\langle t_\alpha'(t), \eta_\alpha(t) \rangle = -y'(t).x''(t) + x'(t).y''(t)$$
$$= \det \begin{pmatrix} x'(t) & x''(t) \\ y'(t) & y''(t) \end{pmatrix}$$

$$\therefore \quad K_\alpha(t) = \frac{\det \begin{pmatrix} x'(t) & x''(t) \\ y'(t) & y''(t) \end{pmatrix}}{(x'(t)^2 + y'(t)^2)^{3/2}}$$, curvatura em coord.

Exemplo: Considere $\alpha : \mathbb{R} \longrightarrow \mathbb{R}^2$ dada por $\alpha(t) = (a.\cos t, b.\operatorname{sen} t)$ para $a, b > 0$. Calcule $K_\alpha(t)$.



Sol.

$$\alpha'(t) = (-a \operatorname{sen} t, b \cos t) \Rightarrow \eta_\alpha(t) = -(b \cos t, a \operatorname{sen} t)$$
$$\alpha''(t) = (-a.\cos t, -b \operatorname{sen} t)$$

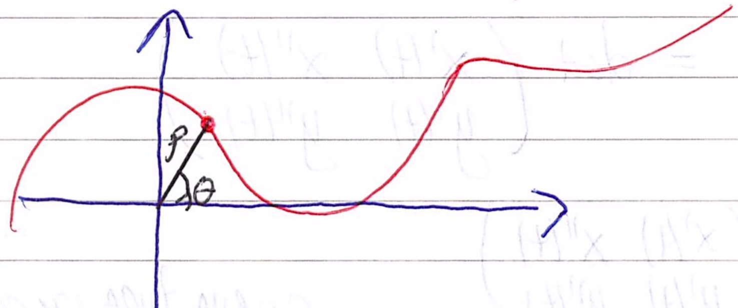
$$\langle \alpha''(t), \eta_\alpha(t) \rangle = a.b \cos^2 t + a.b \operatorname{sen}^2 t$$
$$= a.b$$

$$\|\alpha'(t)\|^3 = (a^2\cos^2 t + b^2 \operatorname{sen}^2 t)^{3/2}$$

$$\therefore \quad K_\alpha(t) = \frac{a \cdot b}{[a^2 \cos^2 t + b^2 \cdot \operatorname{sen}^2 t]^{3/2}}$$

obs.: note que se $a = b = r \Rightarrow K_\alpha(t) = \frac{1}{r}$.

Curvas em coord. polares



Vamos parametrizar $\rho = f(\theta)$. Então

$$\beta(\theta) = (x(\theta), y(\theta)) = (f(\theta) \cdot \cos\theta, f(\theta)\operatorname{sen}\theta).$$

$$= f(\theta) \cdot \vartheta(\theta),$$

no qual $\vartheta(\theta) = (\cos\theta, \operatorname{sen}\theta)$.

Objetivo: encontrar K_β em função de $f(\theta)$.

$$\beta'(\theta) = f'(\theta) \cdot \vartheta(\theta) + f(\theta) \cdot \vartheta'(\theta)$$

$\hookrightarrow (-\operatorname{sen}\theta, \cos\theta)$

$$= f'(\theta) \cdot \vartheta(\theta) + f(\theta) \cdot J\vartheta(\theta).$$

$$\therefore \eta_\beta(\theta) = J(\beta'(\theta)) = f'(\theta) \cdot J\vartheta(\theta) - f(\theta) \cdot \vartheta(\theta)$$

$$\beta''(\theta) = f''(\theta).\,v(\theta) + f'(\theta).\,Jv(\theta)$$
$$+ f'(\theta).\,Jv(\theta) + f(\theta).\,(-v(\theta)).$$
$$= f''(\theta).\,v(\theta) - f(\theta).\,v(\theta) + 2f'(\theta).\,Jv(\theta)$$
$$= (f''-f)(\theta).\,v(\theta) + 2f'(\theta).\,Jv(\theta).$$

Assim

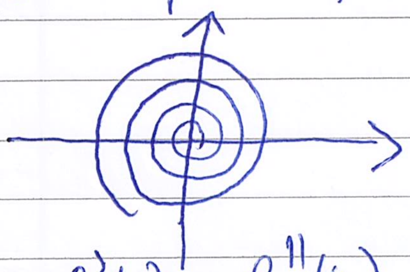
$$\langle \beta''(\theta), n_\beta(\theta)\rangle = \langle f'(\theta).Jv(\theta) - f(\theta).v,\ (f''-f)(\theta).v(\theta) + 2f'(\theta).Jv(\theta)\rangle$$
$$= -f(\theta).(f''-f)(\theta) + 2[f'(\theta)]^2$$

$$\|\beta'(\theta)\|^3 = \left([f'(\theta)]^2 + [f(\theta)]^2\right)^{\frac{3}{2}}$$

Conclusão

$$K_\beta(\theta) = \frac{-f(\theta).(f''-f)(\theta) + 2[f'(\theta)]^2}{([f'(\theta)]^2 + [f(\theta)]^2)^{\frac{3}{2}}}$$

Exemplo: Considere $\beta(\theta) = f(\theta).(\cos\theta, \operatorname{sen}\theta)$, no qual $f(\theta) = e^\theta$



$$f(\theta) = e^\theta = f'(\theta) = f''(\theta) \Rightarrow K_\beta(\theta) = \frac{2e^{2\theta}}{(e^{2\theta} + e^{2\theta})^{\frac{3}{2}}}$$

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Logo $K_\beta(\theta) = \frac{2}{2^{3/2}} \cdot \frac{1}{e^{\theta}} = \frac{1}{\sqrt{2}} \cdot \frac{1}{e^{\theta}}$

EXERCÍCIO: ENCONTRE UMA PARAMETRIZAÇÃO p CA E CALCULE A CURVATURA.