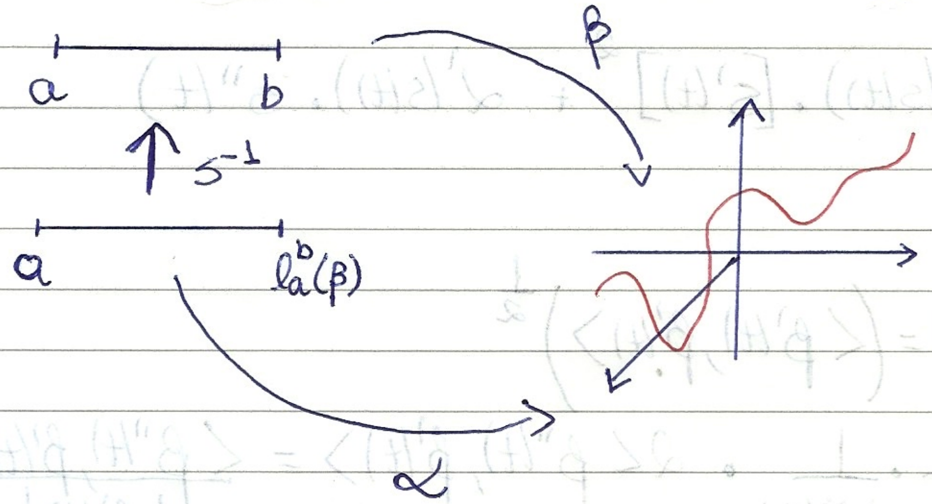
Curvatura e torção de curvas regulares não ppca

Dada $\beta: I \to \mathbb{R}^3$ regular, uma reparametrização $\alpha: I \to \mathbb{R}^3$ de β pelo comprimento do arco é dada por

$$\alpha = \beta \circ s^{-1}, \text{ no qual } s(t) = \int_a^t |\beta'(u)|du$$



Assim $\beta(t) = \alpha(s(t))$, $\forall t \in I$.

Por definição temos

- $t_\beta(t) = t_\alpha(s(t))$
- $\eta_\beta(t) = \eta_\alpha(s(t))$
- $b_\beta(t) = b_\alpha(s(t))$
- $k_\beta(t) = k_\alpha(s(t))$
- $\tau_\beta(t) = \tau_\alpha(s(t))$

Nosso objetivo: escrever as funções/vetores acima independente de $s(t)$, isto é, dependente apenas da parametrização $\beta(t)$

Temos

$$\beta'(t) = \alpha'(s(t)) \cdot s'(t)$$
$$= \alpha'(s(t)) \cdot |\beta'(t)|$$
$$= t_\alpha(s(t)) \cdot |\beta'(t)| \quad \Rightarrow t_\beta(t) = \frac{\beta'(t)}{|\beta'(t)|}$$

Agora

$$\beta''(t) = \alpha''(s(t)) \cdot [s'(t)]^2 + \alpha'(s(t)) \cdot s''(t)$$

$s''(t)$?

$$s'(t) = |\beta'(t)| = \left(\langle \beta'(t), \beta'(t) \rangle\right)^{\frac{1}{2}}$$

$$\Rightarrow s''(t) = \frac{1}{2} \cdot \frac{1}{|\beta'(t)|} \cdot 2\langle \beta''(t), \beta'(t) \rangle = \frac{\langle \beta''(t), \beta'(t) \rangle}{|\beta'(t)|}$$

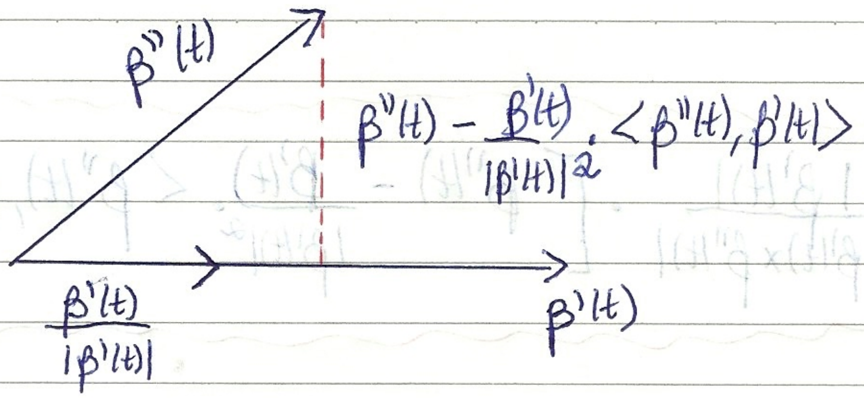
Substituindo na identidade acima temos

$$\alpha''(s(t)) \cdot [s'(t)]^2 = \beta''(t) - \alpha'(s(t)) \cdot s''(t)$$

$$\alpha''(s(t)) \cdot |\beta'(t)|^2 = \beta''(t) - t_\alpha(s(t)) \cdot \frac{\langle \beta''(t), \beta'(t) \rangle}{|\beta'(t)|}$$

$$\alpha''(s(t)) = \frac{\beta''(t)}{|\beta'(t)|^2} - \frac{\beta'(t)}{|\beta'(t)|^4} \cdot \langle \beta''(t), \beta'(t) \rangle$$

$$= \frac{1}{|\beta'(t)|^2} \left[\beta''(t) - \frac{\beta'(t)}{|\beta'(t)|^2} \cdot \langle \beta''(t), \beta'(t) \rangle \right]$$



$$n_\beta(t) = n_\alpha(s(t)) = \frac{\alpha''(s(t))}{|\alpha''(s(t))|} = ?$$ (possível mas muito abstrato)

Note que $K_\beta(t) = K_\alpha(s(t)) = |\alpha''(s(t))|$
↳ definição.

Álgebra Linear: $|v \times w| = |v| \cdot |w| \cdot \operatorname{sen}(\measuredangle(v,w))$

Como $\alpha' \perp \alpha''$ pois α é p.p.c. temos

$$|\alpha'(s(t)) \times \alpha''(s(t))| = |\alpha'(s(t))| \cdot |\alpha''(s(t))| \cdot \underbrace{\operatorname{sen}\frac{\pi}{2}}_{=1}$$

$$\left| \frac{\beta'(t)}{|\beta'(t)|} \times \frac{\beta''(t)}{|\beta'(t)|^2} \right| = |\alpha''(s(t))|$$
↳ pois α é ppca e $|\alpha'(s(t))| = 1$

$$\Rightarrow \quad K_\beta(t) = |\alpha''(s(t))| = \frac{|\beta'(t) \times \beta''(t)|}{|\beta'(t)|^3}$$

Voltando no cálculo de $n_\beta(t)$ temos

$$n_\beta(t) = \frac{\alpha''(s(t))}{|\alpha''(s(t))|} = \frac{|\beta'(t)|^3}{|\beta'(t) \times \beta''(t)|} \cdot \frac{1}{|\beta'(t)|^2} \left[\beta''(t) - \frac{\beta'(t)}{|\beta'(t)|^2} \cdot \langle \beta''(t), \beta'(t) \rangle \right]$$

Logo

$$\eta_\beta(t) = \frac{|\beta'(t)|}{|\beta'(t)\times\beta''(t)|} \cdot \left[\beta''(t) - \frac{\beta'(t)}{|\beta'(t)|^2} \cdot \langle \beta''(t), \beta'(t) \rangle \right]$$

Agora

$$b_\beta(t) = b_\alpha(s(t)) = t_\alpha(s(t)) \times \eta_\alpha(s(t))$$
$$= t_\beta(t) \times \eta_\beta(t)$$
$$= \frac{\beta'(t)\times\beta''(t)}{|\beta'(t)\times\beta''(t)|}$$

Finalmente

$$\tau_\beta(t) = \tau_\alpha(s(t)) = \langle b_\alpha'(s(t)), \eta_\alpha(s(t)) \rangle$$

Agora $b_\beta'(t) = b_\alpha'(s(t)) \cdot s'(t)$
$$= b_\alpha'(s(t)) \cdot |\beta'(t)|$$

$$\Rightarrow \quad \tau_\beta(t) = \left\langle \frac{b_\beta'(t)}{|\beta'(t)|}, \eta_\beta(t) \right\rangle$$

Exercício: $[v(t)\times w(t)]' = v'(t)\times w(t) + v(t)\times w'(t)$

Assim $b_\beta'(t) = \left[\frac{1}{|\beta'(t)\times\beta''(t)|}\right]' \cdot \beta'(t)\times\beta''(t) + \frac{1}{|\beta'(t)\times\beta''(t)|} \cdot [\beta'(t)\times\beta''(t)]'$

$\hookrightarrow$ Regra do produto!

$$= -\,|\beta'(t)\times\beta''(t)|' \cdot \frac{\beta'(t)\times\beta''(t)}{|\beta'(t)\times\beta''(t)|^2} + \frac{[\beta'(t)\times\beta'''(t)]}{|\beta'(t)\times\beta''(t)|}$$

$$= -\,|\beta'(t)\times\beta''(t)|' \cdot \frac{\beta'(t)\times\beta''(t)}{|\beta'(t)\times\beta''(t)|^2} + \frac{\beta'(t)\times\beta'''(t)}{|\beta'(t)\times\beta''(t)|}$$

ALGEBRA LINEAR $\langle v\times w, u\rangle = \det(v,w,u)$

EXERCÍCIO:

$$\tau_\beta(t) = \frac{1}{|\beta'(t)|}\,\langle b_\beta'(t), n_\beta(t)\rangle$$

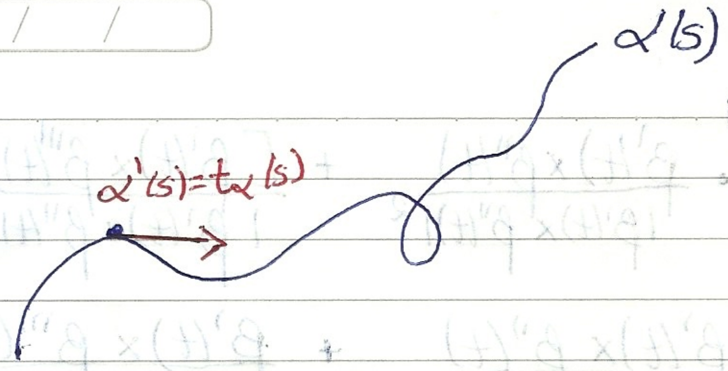
$$= \frac{1}{|\beta'(t)|}\cdot\frac{1}{|\beta'(t)\times\beta''(t)|}\cdot\langle \beta'(t)\times\beta'''(t), \beta''(t)\rangle \cdot \frac{|\beta'(t)|}{|\beta'(t)\times\beta''(t)|}$$

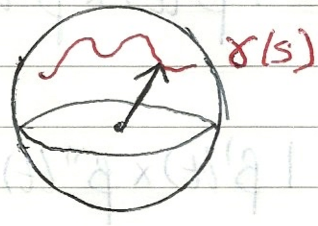
$$= \frac{\langle \beta'(t)\times\beta'''(t), \beta''(t)\rangle}{|\beta'(t)\times\beta''(t)|^2}$$

PELA propriedade de A.L. acima

$$\tau_\beta(t) = \frac{-\det(\beta'(t), \beta''(t), \beta'''(t))}{|\beta'(t)\times\beta''(t)|^2}$$

Exemplo: Seja α uma curva ppca e $\gamma \doteq t_\alpha : I \to S^2 \subset \mathbb{R}^3$ sua indicatriz esférica. Calcule k_γ e τ_γ em termos de k_α e τ_α.





sol. VAMOS USAR s COMO PARAMETRO AQUI

$$\gamma'(s) = t_\alpha'(s) = \kappa_\alpha(s)\cdot\eta_\alpha(s)$$

$$\gamma''(s) = \kappa_\alpha'(s)\cdot\eta_\alpha(s) + \kappa_\alpha(s)\cdot\eta_\alpha'(s)$$

$$= \kappa_\alpha'(s)\cdot\eta_\alpha(s) + \kappa_\alpha(s)\cdot\left[-\kappa_\alpha(s)\,t_\alpha(s) - \tau_\alpha(s)\cdot b_\alpha(s)\right]$$

$$= -[\kappa_\alpha(s)]^2\,t_\alpha(s) + [\kappa_\alpha'(s)]\cdot\eta_\alpha(s) + [-\kappa_\alpha(s)\cdot\tau_\alpha(s)]\cdot b_\alpha(s)$$

$$\gamma'(s)\times\gamma''(s) = [\kappa_\alpha(s)]^3 b_\alpha(s) - [\kappa_\alpha(s)]^2\cdot\tau_\alpha(s)\cdot t_\alpha(s)$$

↳ EXERCÍCIO

$\Rightarrow$

$$|\gamma'(s)\times\gamma''(s)|^2 = [\kappa_\alpha(s)]^6 + [\kappa_\alpha(s)]^4[\tau_\alpha(s)]^2$$

$$= [\kappa_\alpha(s)]^4\cdot\left[[\kappa_\alpha(s)]^2 + [\tau_\alpha(s)]^2\right]$$

PRECISAMOS CALCULAR $\gamma'''(s)$. AO INVÉS DE CALCULAR DERIVANDO $\gamma''(s)$ VAMOS APENAS CALCULAR AS COMPONENTES DO VETOR $\gamma'''(s)$ ÚTEIS PARA COMPUTAR:

$$\langle \gamma'(s)\times\gamma'''(t), \gamma''(t)\rangle$$

Note que

$$\gamma'''(s) = \langle \gamma'''(s), t_\alpha(s)\rangle . t_\alpha(s) + \langle \gamma'''(s), b_\alpha(s)\rangle . b_\alpha(s)$$
$$+ \langle \gamma'''(s), \eta_\alpha(s)\rangle . \eta_\alpha(s)$$

E precisamos apenas determinar

$$x(s) \doteq \langle \gamma'''(s), t_\alpha(s)\rangle$$
$$z(s) \doteq \langle \gamma'''(s), b_\alpha(s)\rangle$$

pois $\gamma'(s)$ é multiplo de $\eta_\alpha(s)$.

Agora

$$\begin{aligned} x(s) &= \langle \gamma''(s), t_\alpha(s)\rangle' - \langle \gamma''(s), t_\alpha'(s)\rangle \\ &= [-[k_\alpha(s)]^2]' - k_\alpha(s). k_\alpha'(s) \\ &= -2k_\alpha(s). k_\alpha'(s) - k_\alpha(s). k_\alpha'(s) \\ &= -3k_\alpha(s). k_\alpha'(s) \end{aligned}$$

$$\begin{aligned} z(s) &= \langle \gamma''(s), b_\alpha(s)\rangle' - \langle \gamma''(s), b_\alpha'(s)\rangle \\ &= [-k_\alpha(s). \tau_\alpha(s)]' - \tau_\alpha(s) . \langle \gamma''(s), \eta_\alpha(s)\rangle \\ &= -k_\alpha'(s). \tau_\alpha(s) - k_\alpha(s). \tau_\alpha'(s) - \tau_\alpha(s). k_\alpha'(s). \\ &= -2k_\alpha'(s). \tau_\alpha(s) - k_\alpha(s). \tau_\alpha'(s). \end{aligned}$$

$$\tau_\gamma(s) = -\frac{\det(\gamma'(s), \gamma''(s), \gamma'''(s))}{[\gamma'(s) \times \gamma''(s)]^2}$$

$$\det \left(\begin{matrix} 0 & -k_\alpha^2 & x \\ k_\alpha & k_\alpha' & y \\ 0 & -k_\alpha \cdot \tau_\alpha & z \end{matrix} \right) \begin{matrix} 0 & -k_\alpha^2 \\ k_\alpha & k_\alpha' \\ 0 & -k_\alpha \tau_\alpha . \end{matrix}$$

$$= -[k_\alpha(s)]^2 \cdot \tau_\alpha(s) x(s) + [k_\alpha(s)]^3 \cdot y(s)$$

$$= [k_\alpha(s)]^3 \cdot [\, k_\alpha(s) \cdot \tau_\alpha'(s) - k_\alpha'(s) \cdot \tau_\alpha(s) \,]$$

Conclusão

$$\tau_\gamma(s) = \frac{+k_\alpha'(s) \cdot \tau_\alpha(s) - k_\alpha(s) \cdot \tau_\alpha'(s)}{k_\alpha(s) \cdot [[k_\alpha(s)]^2 + [\tau_\alpha(s)]^2]}$$

$$k_\gamma(s) = \frac{|\gamma'(t) \times \gamma''(t)|}{|\gamma'(t)|^3} = \frac{[k_\alpha(s)]^2}{[k_\alpha(s)]^3} \cdot \sqrt{[k_\alpha(s)]^2 + [\tau_\alpha(s)]^2}$$

$$\Rightarrow \quad [k_\gamma(s)]^2 = \frac{k_\alpha(s)^2 + \tau_\alpha(s)^2}{k_\alpha(s)^2} .$$

Corolário: α é uma hélice $\Leftrightarrow \dfrac{\tau_\alpha}{k_\alpha} = cte$

$\Leftrightarrow \tau_\gamma = 0$

$\Leftrightarrow \gamma$ é uma curva plana

$\Leftrightarrow \gamma(I)$ está contida em uma circunferência.